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The soliton solutions of the (1+1)-dimensional real ϕ^4 field at finite temperature

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Abstract. We propose a method based on the concepts of coherent state and the approach of the real time Green function to investigate the spontaneous breaking of symmetry in a (1+1)-dimensional ϕ^4 field as well as its restoration at finite temperatures. A critical temperature T_c is found at which the kink-like soliton disappears. In the weak coupling limit, the soliton mass $E_s|_{T=T_c} = (1/3\sqrt{3})E_s|_{T=0}$ and the critical temperature $T_c = (2/3\sqrt{3})E_s|_{T=0}$.

1. Introduction

In a series of papers by two of the present authors, Ni and Su (1980a, b, c), it is argued that the spontaneous breaking of vacuum symmetry in a Higgs field may be viewed as some kind of phase transition of a boson system from the normal state to the superfluidity state. The condensation density of bosons with zero momentum corresponds to the vacuum expectation value of the Higgs field. In discussing these problems, we performed a Bogoliubov transformation

$$\hat{a}_0 = \sqrt{N_0} + \hat{c}_0 \tag{1.1}$$

where $\hat{a}_0$, being the annihilation operator of bosons with zero momenta (see § 2), annihilates the naive vacuum $|0\rangle$ as follows:

$$\hat{a}_0|0\rangle = 0. \tag{1.2}$$

Aiming at finding a new vacuum state $|\tilde{0}\rangle$ which will be annihilated by the new operator $\hat{c}_0$, we have

$$\hat{c}_0|\tilde{0}\rangle = 0. \tag{1.3}$$

Substituting (1.1) into (1.3) we see that

$$\hat{a}_0|\tilde{0}\rangle = \sqrt{N_0}|\tilde{0}\rangle. \tag{1.4}$$

The new vacuum $|\tilde{0}\rangle$, being an eigenstate of an annihilation operator $\hat{a}_0$ with eigenvalue $\sqrt{N_0}$, is a coherent state. N_0 is the occupation number of bosons with zero momentum in this condensation phase. Now let us generalise the above concept

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to the boson condensation with non-zero momenta. In other words, we will try to find a coherent state $|f(\boldsymbol{k})\rangle$ which is annihilated as follows:

$$\hat{c}(\boldsymbol{k})|f(\boldsymbol{k})\rangle = 0. \tag{1.5}$$

By a generalised Bogoliubov transformation

$$\hat{a}(\boldsymbol{k}) = f(\boldsymbol{k}) + \hat{c}(\boldsymbol{k}) \tag{1.6}$$

we have

$$\hat{a}(\boldsymbol{k})|f(\boldsymbol{k})\rangle = f(\boldsymbol{k})|f(\boldsymbol{k})\rangle. \tag{1.7}$$

The property of the coherent state $|f(\boldsymbol{k})\rangle$ is discussed further in appendix 1.

In this paper, we study the $(1+1)$-dimensional real Higgs ϕ^4 field by the above concept. After quantising the Hamiltonian in § 2, we perform the transformation (1.6) in § 3 and find three types of solutions for $f(k)$, one of which describes a soliton. Then in § 4 by using the real time Green function method at zero temperature, we get the elementary excitation spectra in these three cases. In § 5 the above method is generalised to finite temperature situations, and we find that the masses of the soliton as well as the elementary excitations are all temperature dependent. Then in § 6 a critical temperature is found above which the symmetry broken system restores to the original symmetric state accompanying the disappearance of soliton solutions. Section 7 will be a summary and further discussions. In two appendices we list some formulae of the neutral scalar coherent states as well as the real time temperature Green function method.

2. The quantisation of the Hamiltonian

Starting from the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{m_b^2}{4}\phi^2 - \frac{g^2}{4}\phi^4 \tag{2.1}$$

with $m_b^2 > 0$, $g^2 > 0$, we get the Hamiltonian density as

$$\mathscr{H} = \tfrac{1}{2}[\pi^2 + (\partial\phi/\partial x)^2 - \tfrac{1}{2}m_b^2\phi^2 + \tfrac{1}{2}g^2\phi^4]. \tag{2.2}$$

Then we perform the following procedures of canonical quantisation:

$$\phi(x, t) = \sum_k (2L\omega_k)^{-1/2}(\hat{a}_k(t) + \hat{a}_{-k}^{+}(t))\,\mathrm{e}^{\mathrm{i}kx} \tag{2.3}$$

$$\pi(x, t) = \sum_k \mathrm{i}(\omega_k/2L)^{1/2}(\hat{a}_k^{+}(t) - \hat{a}_{-k}(t))\,\mathrm{e}^{-\mathrm{i}kx} \tag{2.4}$$

with the operators obeying the usual equal-time commutation relations

$$[\hat{a}_k(t), \hat{a}_{k'}^{+}(t)] = \delta_{kk'}. \tag{2.5}$$

Notice, however, that

$$\omega_k = (\mu^2 + k^2)^{1/2} \tag{2.6}$$

with a mass μ which is arbitrary at the present stage and will be chosen at our disposal. Substituting (2.3) and (2.4) into (2.2) and bringing the latter in the normal order with respect to the operators $\hat{a}^\dagger$ and $\hat{a}$ we have, e.g.,

$$\phi^2 = :\phi^2: + \sum_k \frac{1}{2L\omega_k} = N_\mu(\phi^2) + \frac{1}{2\pi} \ln \frac{2\Lambda}{\mu}. \tag{2.7}$$

In $:\phi^2: \equiv N_\mu(\phi^2)$, all creation operators $a^\dagger$ stand on the left of the annihilation operators a. Λ is the cut-off of momentum. Then after integrating with respect to x, we get the following Hamiltonian with constant terms discarded:

$$:H: = H_2 + H_4 \tag{2.8}$$

$$H_2 = \sum_k \left[\left(\frac{k^2}{2\omega_k} + \frac{\omega_k}{2} - \frac{m^2}{4\omega_k}\right)(\hat{a}_k^\dagger \hat{a}_k) + \left(\frac{k^2}{4\omega_k} - \frac{\omega_k}{4} - \frac{m^2}{8\omega_k}\right)(\hat{a}_k^\dagger \hat{a}_{-k}^\dagger + \hat{a}_k \hat{a}_{-k})\right] \tag{2.9}$$

$$\begin{aligned} H_4 = \frac{g^2}{16L} \sum_{k_1k_2k_3k_4} \frac{\delta_{k_1+k_2+k_3+k_4,0}}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}} [&\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4} + \hat{a}_{-k_1}^\dagger \hat{a}_{-k_2}^\dagger \hat{a}_{-k_3}^\dagger \hat{a}_{-k_4}^\dagger \\ &+ 4(\hat{a}_{-k_1}^\dagger \hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4} + \hat{a}_{-k_1}^\dagger \hat{a}_{-k_2}^\dagger \hat{a}_{-k_3}^\dagger \hat{a}_{k_4}) + 6\hat{a}_{-k_1}^\dagger \hat{a}_{-k_2}^\dagger \hat{a}_{k_3}\hat{a}_{k_4}] \end{aligned} \tag{2.10}$$

with

$$m^2 = m_b^2 - (3g^2/\pi) \ln (2\Lambda/\mu) \tag{2.11}$$

being the renormalised parameter. Obviously, m^2 is μ dependent. The coupling constant g^2 does not change in this renormalisation. The fact that the normal ordering is equivalent to a renormalisation was demonstrated by Coleman (1975) in the sine-Gordon system.

3. The generalised Bogoliubov transformation

Notice that

$$\hat{a}_k \leftrightarrow (2\pi/L)^{1/2} \hat{a}(k) \tag{3.1}$$

and perform the generalised Bogoliubov transformation

$$\hat{a}(k) = f(k) + \hat{c}(k); \tag{3.2}$$

then we are able to recast the Hamiltonian (2.8) into the following form:

$$:H: = H_0' + H_1' + H_2' + H_3' + H_4' \tag{3.3}$$

where

$$\begin{aligned} H_0' = \int \Bigg\{ &\left[\frac{1}{2}\left(\frac{k^2}{\omega_k} + \omega_k\right) - \frac{m^2}{4\omega_k}\right] f^*(k) f(k) \\ &+ \left[\frac{1}{2}\left(\frac{k^2}{2\omega_k} - \frac{\omega_k}{2}\right) - \frac{m^2}{8\omega_k}\right] [f(k)f(-k) + f^*(k)f^*(-k)]\Bigg\} \, \mathrm{d}k \\ &+ \frac{g^2}{32\pi} \int \frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}} \{f(k_1)f(k_2)f(k_3)f(k_4) \\ &+ f^*(-k_1)f^*(-k_2)f^*(-k_3)f^*(-k_4) \\ &+ 4[f^*(-k_1)f(k_2)f(k_3)f(k_4) + f^*(-k_1)f^*(-k_2)f^*(-k_3)f(k_4)] \\ &+ 6f^*(-k_1)f^*(-k_2)f(k_3)f(k_4)\} \, \mathrm{d}k_1 \, \mathrm{d}k_2 \, \mathrm{d}k_3 \, \mathrm{d}k_4 \end{aligned} \tag{3.4}$$

$$H_1' = \int \left[\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right) - \frac{m^2}{4\omega_k}\right][f(-k)\hat{c}^\dagger(-k)+f^*(k)\hat{c}(k)]\,\mathrm{d}k + \int \left[\left(\frac{k^2}{2\omega_k}-\frac{\omega_k}{2}\right)-\frac{m^2}{4\omega_k}\right]$$
$$\times[f(-k)\hat{c}(k)+f^*(k)\hat{c}^\dagger(-k)]\,\mathrm{d}k + \frac{g^2}{8\pi}\int \frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}$$
$$\times\{[f(k_1)f(k_2)f(k_3)+3f^*(-k_1)f(k_2)f(k_3)+f^*(-k_1)f^*(-k_2)f^*(-k_3)$$
$$+3f^*(-k_1)f^*(-k_2)f(k_3)][\hat{c}(k_4)+\hat{c}^\dagger(-k_4)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 \tag{3.5}$$

$$H_2' = \int \left\{\left[\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right)-\frac{m^2}{4\omega_k}\right]\hat{c}^\dagger(k)\hat{c}(k)+\left[\frac{1}{2}\left(\frac{k^2}{2\omega_k}-\frac{\omega_k}{2}\right)-\frac{m^2}{8\omega_k}\right]\right.$$
$$\left.\times[\hat{c}(k)\hat{c}(-k)+\hat{c}^\dagger(k)\hat{c}^\dagger(-k)]\right\}\mathrm{d}k + \frac{3g^2}{16\pi}\int\frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}$$
$$\times\{[f(k_1)f(k_2)+2f^*(-k_1)f(k_2)+f^*(-k_1)f^*(-k_2)][\hat{c}(k_3)\hat{c}(k_4)$$
$$+\hat{c}^\dagger(-k_3)\hat{c}^\dagger(-k_4)+2\hat{c}^\dagger(-k_3)\hat{c}(k_4)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 \tag{3.6}$$

$$H_3' = \frac{g^2}{8\pi}\int\frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}\{[f(k_1)+f^*(-k_1)][\hat{c}(k_2)\hat{c}(k_3)\hat{c}(k_4)$$
$$+\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}^\dagger(-k_4)+3\hat{c}^\dagger(-k_2)\hat{c}(k_3)\hat{c}(k_4)$$
$$+3\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}(k_4)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 \tag{3.7}$$

$$H_4' = \frac{g^2}{32\pi}\int\frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}\{\hat{c}(k_1)\hat{c}(k_2)\hat{c}(k_3)\hat{c}(k_4)$$
$$+\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}^\dagger(-k_4)$$
$$+4[\hat{c}^\dagger(-k_1)\hat{c}(k_2)\hat{c}(k_3)\hat{c}(k_4)+\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}(k_4)]$$
$$+6\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}(k_3)\hat{c}(k_4)\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{3.8}$$

At zero temperature, we assume that the ground state is simulated by a coherent state $|f\rangle$ which is characterised by a momentum distribution function $f(k)$ as shown in (3.2). $f(k)$ should be determined by the variation conditions

$$\delta\langle{:}H{:}\rangle/\delta f(p) = \delta H_0'/\delta f(p) = 0 \tag{3.9}$$

and

$$\delta\langle{:}H{:}\rangle/\delta f^*(p) = \delta H_0'/\delta f^*(p) = 0. \tag{3.10}$$

Thus we have

$$f(-p) = f^*(p) \tag{3.11}$$

and

$$\left(\frac{p^2}{\omega_p}-\frac{m^2}{2\omega_p}\right)f(p)+\frac{g^2}{\pi}\int\frac{f(k_1)f(k_2)f(p+k_1+k_2)}{(\omega_{k_1}\omega_{k_2}\omega_p\omega_{p+k_1+k_2})^{1/2}}\,\mathrm{d}k_1\,\mathrm{d}k_2 = 0. \tag{3.12}$$

Equation (3.12) can also be derived from the condition

$$H_1' = 0. \tag{3.13}$$

Introduction of

$$y(p) = f(p)/\sqrt{\omega_p} \tag{3.14}$$

leads to

$$(p^2-\tfrac{1}{2}m^2)y(p)+(g^2/\pi)\int \mathrm{d}k_1\,\mathrm{d}k_2\,y(k_1)y(k_2)y(p+k_1+k_2)=0. \quad (3.15)$$

Furthermore, we carry out a Fourier transformation

$$y(p)=\int_{-\infty}^{\infty}\tilde{y}(u)\,\mathrm{e}^{-\mathrm{i}pu}\,\mathrm{d}u \quad (3.16)$$

and get an equation for $\tilde{y}(u)$ as

$$-\mathrm{d}^2\tilde{y}(u)/\mathrm{d}u^2-\tfrac{1}{2}m^2\tilde{y}(u)+4\pi g^2\tilde{y}^3(u)=0. \quad (3.17)$$

There are three types of solutions for equation (3.17):

(a)
$$\tilde{y}(u)=0 \quad (3.18)$$
$$f(k)=0. \quad (3.19)$$

The expectation value of energy in vacuum reads as

$$\langle :H:\rangle \equiv U=U(a)=0. \quad (3.20)$$

This trivial solution implies that our system remains in the unstable normal state.

(b)
$$\tilde{y}(u)=\pm m/2\sqrt{2\pi}g. \quad (3.21)$$

Thus

$$f_{\mathrm{even}}(k)=\pm(m/g)(\tfrac{1}{2}\pi\omega_0)^{1/2}\delta(k) \quad (3.22)$$
$$\langle\phi\rangle=\pm m/\sqrt{2}g \quad (3.23)$$
$$U(b)=U_{\mathrm{even}}=-\tfrac{1}{16}m^4L/g^2. \quad (3.24)$$

Obviously, this solution corresponds to a spontaneously symmetry broken state in which the boson condensation with zero momentum occurs.

(c)
$$y(u)=\pm(m/2\sqrt{2\pi}g)\tanh\tfrac{1}{2}mu. \quad (3.25)$$

Thus

$$f_{\mathrm{odd}}(k)=\mp\mathrm{i}g^{-1}(\tfrac{1}{2}\pi\omega_k)^{1/2}\,\mathrm{cosech}(\pi k/m) \quad (3.26)$$
$$\langle\phi\rangle=\pm(m/\sqrt{2}g)\tanh\tfrac{1}{2}mx \quad (3.27)$$
$$U(c)=U_{\mathrm{odd}}=-\tfrac{1}{16}m^4L/g^2+m^3/3g^2. \quad (3.28)$$

This is a well known kink-like soliton in configuration space which implies that the Bose condensation not only occurs at zero momentum but prevails over the whole range of momenta. The last term in (3.28) is just the mass of the soliton at zero temperature:

$$E_{\mathrm{s}}|_{T=0}=m^3/3g^2. \quad (3.29)$$

4. The Green function method (zero temperature)

We shall use the method of the equation of motion for the real time Green function to find out the elementary excitation spectrum of our boson system. Similar treatment has been carried out for some fermion systems (Ni 1983). The formulae used in this paper are listed in appendix 2.

Working in lowest approximation, we define the momentum conserved Green function as

$$G_1(p,p) \equiv G_1 = \langle\langle \hat{c}_p(t) | c_p^\dagger(t') \rangle\rangle \tag{4.1}$$

$$G_2(p,p) \equiv G_1 = \langle\langle c_{-p}^\dagger(t) | c_p^\dagger(t') \rangle\rangle. \tag{4.2}$$

Then the motion equations of these two Green functions can be written in spectral representation as

$$E\tilde{G}_1 = (2\pi)^{-1} + \Omega_p \tilde{G}_1 + \Delta_p \tilde{G}_2 \qquad E\tilde{G}_2 = -\Delta_p \tilde{G}_1 - \Omega_p \tilde{G}_2 \tag{4.3}$$

where

$$\tilde{G}_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, G_j(t,t') \exp[\mathrm{i}E(t,t')] \qquad (j=1,2) \tag{4.4}$$

and

$$\Omega_p = \left(\frac{p^2}{2\omega_p} + \frac{\omega_p}{2} - \frac{m^2}{4\omega_p}\right) + \frac{3g^2}{L\omega_p} \int \frac{1}{\omega_k} f(k) f(-k)\, \mathrm{d}k \qquad \Delta_p = \Omega_p - \omega_p. \tag{4.5}$$

In deriving the algebraic equation (4.3), we have made a pairing approximation, i.e. a factor $(2\pi/L)\delta(k_1+k_2)$ has been plunged into H_2'. At the same time, we neglect all the contributions of H_3' and H_4' because of the fact that $\langle \hat{c}_k^\dagger \hat{c}_k \rangle = \langle \hat{c}_k^\dagger \hat{c}_{-k}^\dagger \rangle = 0$ at zero temperature. The solution of equation (4.3) is well known:

$$\tilde{G}_1 = \frac{1}{2\pi} \frac{E+\Omega_p}{E^2 - (\Omega_p^2 - \Delta_p^2)} \qquad \tilde{G}_2 = -\frac{1}{2\pi} \frac{\Delta_p}{E^2 - (\Omega_p^2 - \Delta_p^2)} \tag{4.6}$$

which in turn gives us the elementary excitation spectrum of the system as

$$E_p^2 = \Omega_p^2 - \Delta_p^2 = p^2 - \frac{m^2}{2} + \frac{6g^2}{L} \int \frac{\mathrm{d}k}{\omega_k} f(k) f(-k). \tag{4.7}$$

If our system is in uniform condensation phase as shown by (3.22), we find

$$E_p^2 = p^2 + m^2. \tag{4.8}$$

However, if there is a soliton background as shown by (3.26), the calculation leads to†

$$E_p^2 = p^2 + m^2 - 6m/L \tag{4.9}$$

which is essentially the same value of (4.8) as $L \to \infty$. So in either case we learn that the mass of elementary excitations equals m.

5. The ground state of the system at finite temperature

We are now in a position to consider what would happen at finite temperatures. First of all, we replace the vacuum average by the ensemble average but with the same

† $\displaystyle\int_{-\infty}^{\infty} \operatorname{cosech} a(p+k) \operatorname{cosech} ak\, \mathrm{d}k = \left(\frac{\pi^2}{a^2}\delta(p) - 2p \operatorname{cosech} ap\right) \xrightarrow[p\to 0]{} \frac{\pi L}{2a^2} - \frac{2}{a}.$

symbol. In this case we make every possible pairing of momenta in H'_3 and reduce it to

$$\tilde{H}'_3 = 6g^2\nu \int \mathrm{d}k\, \omega_k^{-1} f(-k)[\hat{c}(k)+\hat{c}^+(-k)] \tag{5.1}$$

where a dimensionless parameter ν has been picked out,

$$\nu = \sum_k \frac{1}{2\omega_k L}\langle \hat{c}^+_k\hat{c}_k + \hat{c}^+_k\hat{c}^+_{-k}\rangle = \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{\omega_k}\langle \hat{c}^+_k\hat{c}_k + \hat{c}^+_k\hat{c}^+_k\rangle. \tag{5.2}$$

ν is temperature dependent and approaches zero when $T \to 0$. Then the condition

$$\tilde{H}'_1 \equiv H'_1 + \tilde{H}'_3 = 0 \tag{5.3}$$

will lead to an equation for $f(p)$ which can also be derived from the variation principle

$$\delta\langle :H:\rangle/\delta f(p) = 0 \tag{5.4}$$

with

$$\langle :H:\rangle = H_0 + \langle \tilde{H}'_2\rangle + \langle \tilde{H}'_4\rangle. \tag{5.5}$$

In $\langle \tilde{H}'_2\rangle$ and $\langle \tilde{H}'_4\rangle$ the same pairing approximation has been made. Equation (5.4) reads as

$$-\mathrm{d}^2\tilde{y}(u)/\mathrm{d}u^2 - \tfrac{1}{2}M^2\tilde{y}(u) + 4\pi g^2\tilde{y}^3(u) = 0 \tag{5.6}$$

where

$$M^2 = m^2 - 12g^2\nu. \tag{5.7}$$

Comparison between (5.6) and (3.17) is quite interesting. The only change is ascribed to replacing m to M. Under the same approximation, all the expressions from (3.18) through (4.9) remain valid except the substitution of M for m. In particular, the elementary excitation (phonon) spectrum in case (b) or (c) is

$$E_p^2 = p^2 + M^2 \tag{5.8}$$

while the mass of the soliton becomes

$$U(c) - U(b) = U_{\mathrm{odd}} - U_{\mathrm{even}} = M^3/3g^2. \tag{5.9}$$

Notice, however, that $U(a)$, $U(b)$ or $U(c)$ themselves will receive an extra contribution from the thermal excitation of phonons (see below).

6. The critical temperature and phase transition

Let us focus our attention on the parameter ν. Using the formulae (see appendix 2)

$$\langle \hat{c}^+_p\hat{c}_p\rangle = \mathrm{i}\int_{-\infty}^{\infty} \frac{\tilde{G}_1(E+\mathrm{i}\varepsilon) - \tilde{G}_1(E-\mathrm{i}\varepsilon)}{\mathrm{e}^{\beta E}-1}\,\mathrm{d}E \tag{6.1}$$

$$\langle \hat{c}^+_p\hat{c}^+_{-p}\rangle = \mathrm{i}\int_{-\infty}^{\infty} \frac{\tilde{G}_2(E+\mathrm{i}\varepsilon) - \tilde{G}_2(E-\mathrm{i}\varepsilon)}{\mathrm{e}^{\beta E}-1}\,\mathrm{d}E \tag{6.2}$$

$$(E - E_p \pm \mathrm{i}\varepsilon)^{-1} = P(E-E_p)^{-1} \mp \mathrm{i}\pi\delta(E-E_p) \tag{6.3}$$

we find

$$\nu = \frac{1}{4\pi} \int_0^\infty \frac{dp}{\omega_p} \left[\frac{(1+\omega_p/E_p)}{\exp(\beta E_p) - 1} + \frac{(1-\omega_p/E_p)}{\exp(-\beta E_p) - 1} \right]. \tag{6.4}$$

It is time to take advantage of the arbitrariness of μ in $\omega_p = (p^2 + \mu^2)^{1/2}$. We choose $\mu = M$ to make $\omega_p = E_p = (p^2 + M^2)^{1/2}$ and

$$\nu = \frac{1}{2\pi} \int_0^\infty \frac{dp}{(p^2 + M^2)^{1/2} \{\exp[(p^2 + M^2)^{1/2}/T] - 1\}}. \tag{6.5}$$

Notice that this is the only choice which is consistent with the demand $\nu \xrightarrow[T \to 0]{} 0$ and at the same time we have

$$\langle \hat{c}_p^+ \hat{c}_{-p}^+ \rangle = 0$$

and

$$\langle \hat{c}_p^+ \hat{c}_p \rangle = [\exp(\beta E_p) - 1]^{-1} \tag{6.6}$$

which implies that at a certain temperature the elementary excitations (phonons) obey the stationary Bose–Einstein distribution, a nice property we would like to have from the beginning. We note in passing that a choice of $\mu = 0$ would lead to infrared divergence in the integral (6.5).

The integration of (6.5) can be performed approximately with the following result (Dolan and Jackiw 1974):

$$\nu = (2\pi)^{-1} [\pi T/2M + \tfrac{1}{2} \ln (M/4\pi T) + \tfrac{1}{2}\gamma + O(M^2/T^2)] \tag{6.7}$$

where $\gamma = 0.5772$ is the Euler constant. Substituting (6.7) into (5.7), we get

$$M^2 = m^2 - 3g^2 T/M - (3g^2/\pi)[\ln (M/4\pi T) + \gamma + O(M^2/T^2)]. \tag{6.8}$$

However, we should not forget that m^2 itself, as defined by (2.11), is μ dependent. Since $\mu = M(T)$, m^2 is also a function of temperature. But we can get rid of this uncertainty by defining the renormalised m at zero temperature by

$$m^2|_{T=0} = m_0^2 \tag{6.9}$$

and noting that

$$M^2|_{T=0} = m_0^2. \tag{6.10}$$

Thus

$$M^2(T) = m_0^2 - 3g^2 T/M - (3g^2/\pi)[\ln (m_0/4\pi T) + \gamma + O(M^2/T^2)] \tag{6.11}$$

so $M(T)$ can be expressed as a function of m_0, g and T only. Alternatively, we have

$$M^3 - \{m_0^2 - (3g^2/\pi)[\ln (m_0/4\pi T) + \gamma]\} M + 3g^2 T = 0. \tag{6.12}$$

Then the condition of the existence in this equation of two real roots, one of which is positive, leads to a critical temperature

$$T_c \simeq (2m_0^3/9\sqrt{3}g^2)\{1 - (9g^2/2\pi m_0^2)[\ln (g^2/m_0^2) + 0.0996]\} \tag{6.13}$$

where the weak coupling condition

$$g^2 \ll m_0^2 \tag{6.14}$$

has been used. At this temperature, the mass square of phonons, M^2, equals

$$M^2|_{T=T_c} \simeq \tfrac{1}{3}m_0^2 \tag{6.15}$$

which implies that the soliton mass

$$E_s = M^3/3g^2 \tag{6.16}$$

decreases to $1/3\sqrt{3}$ times that at zero temperature before it disappears at $T = T_c$. When $T > T_c$, M turns to complex, both the condensating phases (b) and (c) cannot exist at all and only the uncondensating phase (a) survives. This is an example of symmetry restoration which is well known since the pioneer work of Kirzhnits and Linde (1972).

We may also calculate the phonon spectrum in the normal phase (a) above the critical temperature. The results are as follows:

$$E_p^2 = p^2 + M'^2 \tag{6.17}$$

where

$$M'^2 \simeq -\tfrac{1}{2}m_0^2 + 3g^2T/2M' \tag{6.18}$$

$$M'^2|_{T=T_c} \simeq 0.1022m_0^2. \tag{6.19}$$

Comparison between (6.15) and (6.19) reveals an abrupt change in phonon spectrum at $T = T_c$. There is also an abrupt change in the energy of the system

$$U(a)|_{T=T_c} - U(b)|_{T=T_c} \simeq (\tfrac{3}{8}MTL)_{T=T_c} = m_0^4L/36g^2 \tag{6.20}$$

so there is a phase transition accompanying the symmetry restoration.

7. Summary and discussion

We present an example of spontaneous breaking of symmetry and its restoration at higher temperatures. Both the symmetry broken state and the soliton state are treated as coherent states in which some kind of particle condensation takes place. At first sight, it is incompatible with the usual assertion that there is no condensation in a $(1+1)$-dimensional boson system at $T \neq 0$ (Hohenberg 1967). But the latter assertion is based on the conservation of particle number which is not valid in the present model. Actually, particles are created from the naive vacuum in this ϕ^4 model.

There are several kinds of methods for handling the quantum field theory at finite temperatures. The approaches developed by Dolan and Jackiw (1974), Weinberg (1974) and Bernard (1974) are based on the loop expansion in calculating the effective potential and the Matsubara imaginary time Green function method. Recently, Maki and Takayama (1979a, b; Takayama and Maki 1979) used essentially the same method to investigate various one-dimensional systems. They got the critical temperature T_c of the ϕ^4 system precisely coinciding with our result (6.13) in the leading term.

Instead of imaginary time, we propose the real time Green function approach in its lowest approximation and find our results at the cost of simple calculation. However, the μ normal-ordering and renormalisation scheme is emphasised to make the whole thing self-consistent which will be discussed further in a subsequent paper (Chen and Ni 1983) where an improved version of this method will be presented.

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Appendix 1. The coherent state of a neutral scalar field

For a neutral scalar field, a coherent state including many particle condensation with changeable momenta can be defined as

$$|f\rangle = N\sum_{n=0}^{\infty}\frac{1}{n!}|n\rangle_f \tag{A1.1}$$

where

$$|n\rangle_f = \left(\int \mathrm{d}\boldsymbol{k}\, f(\boldsymbol{k})\hat{a}^{+}(\boldsymbol{k})\right)^n|0\rangle \tag{A1.2}$$

and

$$[\hat{a}(\boldsymbol{k}), \hat{a}^{+}(\boldsymbol{k}')] = \delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{A1.3}$$

It can be easily proved that

$$\hat{a}(\boldsymbol{p})|f\rangle = f(\boldsymbol{p})|f\rangle. \tag{A1.4}$$

The normalisation condition

$$\langle f|f\rangle = 1 \tag{A1.5}$$

leads to

$$N = \exp(-\tfrac{1}{2}\|f\|^2) = \exp(-\tfrac{1}{2}\bar{n}) \tag{A1.6}$$

where

$$\bar{n} = \langle n\rangle = \int \mathrm{d}\boldsymbol{k}\,\langle f|\hat{a}^{+}(\boldsymbol{k})\hat{a}(\boldsymbol{k})|f\rangle = \int \mathrm{d}\boldsymbol{k}\, f^*(\boldsymbol{k})f(\boldsymbol{k}) = \|f\|^2 \tag{A1.7}$$

is the average number of particles in the coherent state.

Appendix 2. The real time temperature Green function

The Green function composed of two operators $\hat{A}(t)$ and $\hat{B}(t')$ is defined as

$$G_{AB}(t,t') \equiv \langle\langle \hat{A}(t)|\hat{B}(t')\rangle\rangle = -\mathrm{i}\langle T\hat{A}(t)\hat{B}(t')\rangle \tag{A2.1}$$

where

$$T\hat{A}(t)\hat{B}(t') = \theta(t-t')\hat{A}(t)\hat{B}(t') + \eta\theta(t'-t)\hat{B}(t')\hat{A}(t) \tag{A2.2}$$

with $\eta = 1$ when $\hat{A}$, $\hat{B}$ are boson operators, and $\eta = (-1)^P$ when $\hat{A}$, $\hat{B}$ are fermion operators, P is the number of permutations which bring the product $\hat{A}\hat{B}$ to $\hat{B}\hat{A}$.

In (A2.1) the symbol $\langle\ \rangle$ means either the vacuum average at zero temperature or the ensemble average at finite temperatures.

The motion equation for the Green function G_{AB} can be derived straightforwardly as

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)G_{AB}(t-t') = \delta(t-t')\langle[\hat{A}(t), \hat{B}(t')]_{\pm}\rangle + \langle\langle[\hat{A}(t), \hat{H}(t)]|\hat{B}(t')\rangle\rangle \tag{A2.3}$$

where the plus or minus sign in the first term is used for fermion or boson operators respectively. Usually a Fourier transformation is made:

$$\tilde{G}_{AB}(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \exp[\mathrm{i}(t-t')E]G_{AB}(t, t'). \tag{A2.4}$$

Then the time correlation function of the operator product $\langle\hat{B}(t')\hat{A}(t)\rangle$ can be proved as

$$F_{BA}(t, t') \equiv \langle\hat{B}(t')\hat{A}(t)\rangle = \mathrm{i}\int_{-\infty}^{\infty} \frac{\tilde{G}_{AB}(E+\mathrm{i}\varepsilon) - \tilde{G}_{AB}(E-\mathrm{i}\varepsilon)}{\mathrm{e}^{\beta E} - \eta} \exp[-\mathrm{i}E(t-t')]\, \mathrm{d}E \tag{A2.5}$$

where

$$\beta \equiv 1/k_{\mathrm{B}}T = 1/T. \tag{A2.6}$$

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